1 Problems

1. Let D_4 be the dihedral group of symmetries of a square.
   (a) How many elements does D_4 have?
   (b) How many of these are rotations?
   (c) How many are reflections?

2. Find all subgroups of D_4. Which of them are cyclic?

3. Let S_3 be the symmetric group on 3 elements. Let $\sigma = (1\ 2)$ and $\tau = (1\ 2\ 3)$.
   (a) Compute $\sigma\tau$ and $\tau\sigma$ and compare.
   (b) List all subgroups of S_3. Identify which are cyclic and which are not.

4. Prove that set of all rotations in D_n form a subgroup of D_n.

5. Let G be a group and H_1 and H_2 be two subgroups of G.
   (a) Prove that $H_1 \cap H_2$ is a subgroup and also $\bigcap H_\alpha$, $\alpha \in I$ where I is an indexing set
   (b) what about $H_1 \cup H_2$?

6. Verify that $GL_n(\mathbb{R})$ satisfies all four group axioms under matrix multiplication. Define the special linear group $SL_2(\mathbb{R})$ as the set of 2×2 real matrices with determinant 1.
   (a) Show that $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.
   (b) Verify whether the matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ belongs to $SL_2(\mathbb{R})$.

7. Let $\mathbb{Z}$ be the group of integers and let $k, l \in \mathbb{Z}$ .What is the subgroup generated by k and l?

8. Is it possible that the subgroup generated by 2 and 3 in $\mathbb{Q}$ under addition is cyclic?
   (a) What about the subgroup generated by any two elements in $\mathbb{Q}$?
   (b) What about subgroup generated by 2 and 3 in $\mathbb{Q} - \{0\}$ under multiplication?

9. Define $det : GL_n(\mathbb{R}) \to \mathbb{R}$ as the determinant map. Show that this is a group homomorphism-

   (a) What is the kernel of this homomorphism?

   (b) What is its image?

10. Let $H =< P_\sigma; \sigma \in S_n >$ is a subgroup of S_n. Let ϕ is a map from S_n to $GL_n(\mathbb{R})$ defined as $\phi(\sigma) = P_\sigma$. Prove that ϕ is one to one homomorphism.

11. Verify that $GL_n(\mathbb{R})$ satisfies all four group axioms under matrix multiplication. Define the special linear group $SL_2(\mathbb{R})$ as the set of 2×2 real matrices with determinant 1.

   (a) Show that $SL_2(\mathbb{R})$ is a subgroup of $GL_2(\mathbb{R})$.

   (b) Verify whether the matrix $B = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$ belongs to $SL_2(\mathbb{R})$.