Problems 1

1. **Vector Space Identification:** For each of the following, determine whether the given set V forms a vector space over the specified field F with the defined operations.

   (a) $V = \{A \in M_{2\times 2}(\mathbb{R}) \mid \det(A) \neq 0\}$ (all 2×2 invertible matrices with real entries)

   (b) $V = P_n(\mathbb{R})$ (all polynomials with real coefficients of degree *exactly* n, where $n \geq 1$)

   (c) $V = \{(x, y, z) \in \mathbb{R}^3 \mid y = 2x \text{ and } z = 3x\}$ (the set of vectors in $\mathbb{R}^3$ that lie on the line passing through the origin defined by $y = 2x$ and $z = 3x$)

2.
   - **Statement P:** $\forall v \in V, \exists v' \in V$ such that $v + v' = \theta$.
   - **Statement Q:** $\exists v' \in V$ such that $\forall v \in V, v + v' = \theta$.

   Is statement **P** and **Q** are equivalent?

3. Prove the following

   (a) Additive inverse is unique.

   (b) Scalar zero times any vector is zero vector.$(0 * v = \theta)$

   (c) Zero vector is unique.

   (d) Scalar times the zero vector is Zero vector.$(\alpha * \theta = \theta)$

   (e) $(-1) * v = v'$

   (f) $V + V = 2V$.

   (g) $\alpha v = \theta$ if and only if $\alpha = 0$ or $v = \theta$.

Problems 2

1. $\alpha v = 0$ where v is non zero vector then prove $\alpha = 0$.

2. If W is closed under scalar multiplication and it's non empty, then prove $0 \in W$.

3. If W is a non empty set and for any arbitrary $w_1, w_2 \in W$ then $w_1 + \alpha w_2 \in W$, is W is a subspace?

4. $W = \{(x, y) \in \mathbb{R}^2 \mid ax + by = 0\}$, is W is a subspace of $\mathbb{R}^2$?

5. Let $V = \mathbb{R}^n$. Is $W_{(a_1, a_2, \ldots, a_n)} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid a_1 x_1 + \cdots + +a_n x_n = 0\}$ a subspace of V, where $a_i \in \mathbb{R}$ for $i = 1, \ldots, n$?

6. Let $\{V_\alpha\}_{\alpha \in I}$ be an arbitrary collection of subspaces of a vector space V. Is their intersection, $\bigcap_{\alpha \in I} V_\alpha$, also a subspace of V?

7. $X \neq \phi$, $V = F(X, R)$ for $x \in X$, $W_x = \{f \in V/f(x) = 0\}$ is a subspace of V? If $A \subset X$, then $W_A = \bigcap_{a \in A} W_a$. Is W_A a subspace of V? What is W_A?

8. (a) W_1, W_2 two subspace of V, is $W_1 \cup W_2$ a subspace of V?

   (b) Prove that if $W_1 \cup W_2$ is subspace, then $W_1 \in W_2$ or $W_2 \in W_1$.

9. Let $V = \mathbb{R}[x]$ be the vector space of all polynomials with coefficients in $\mathbb{R}$. Let $W = \{f \in V/f(0) = 0\}$. Is W a subspace of V?

10. Is following are subspace of $M_n(\mathbb{R})$

    (a) $W_1 = \{A \in M_n(\mathbb{R}) \mid A = A^T\}$

    (b) $W_2 = \{A \in M_n(\mathbb{R}) \mid A = -A^T\}$

11. What is $W_1 + W_2$?

Problems 3

1. Let $W \subset V$ and $\{v_1, .., v_n\} \subset V$. Then the following are equivalent
   (a) $W = LS(v_1, .., v_n)$
   (b) $v_1, .., v_n \in W$ and if $v_1, .., v_n \in V_1 < V$, then $W \subset V_1$. (The smallest subspace containing $v_1, .., v_n$)
   (c) $W = \cap\{X \mid X < V, v_1, .., v_n \in X\}$
2. Find the dimention of following vector spaces
   (a) $\{a_0 + a_1x^2 + ... + a_nx^n = 0 \mid a_i \in \mathbb{R}\}$
   (b) $\{(a_{ij})_{m*n} \mid a_{k1} + ... + a_{kn} = 0, k = 1, 2, ..., m\}$
   (c) $\{(a_{ij})_{m*n} \mid a_{ij} \in \mathbb{R}, \forall i \neq j, a_{ij} = 0\}$
   (d) $\{(a_ij)_{m*n} \mid tr(a_{ij}) = 0\}$